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Characterization of free fall paths by a global or local Desargues property

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Abstract. *Usually, the paths of free particles in classical mechanics, special relativity, or general relativity are defined by stating that there exist coordinate systems, in which these paths globally or locally obey linear equations. Here, a coordinate-independent characterization of these paths solely by elementary and operative incidence properties is given. It is shown that in 2 dimensions the Pappus figure (with 9 paths and 9 incidence points), and in 3 and more dimensions the Desargues figure (with 10 paths and 10 points) are the simplest possible incidence figures for characterizing the straight paths of an inertial structure. For the case of general relativity, a suitable local modification of the Desargues property is formulated: In every* ϵ*-neighbourhood of every point, the incidence points and the connecting paths of the Desargues figure exist up to corrections of order* ϵ^3 *. It is then proven that this local Desargues property uniquely characterizes the (projective) geodesics of general relativity.*

1. INTRODUCTION

The paths of «free particles» constitute the basis (the kinematic frame) for all of physics, as is especially evident from the foundations of classical mechanics, special relativity, and general relativity. Usually, these paths are introduced by stating that there exist (global or local) coordinate systems $\{x^a\}$ and parametrizations τ such that all of these paths $x^a(\tau)$ obey the differential equation $\frac{d^2 x^a}{d\tau^2} = 0$. It should, however, be clear that such a coordinate-dependent characterization of the basis elements of physics is in no way satisfying, and it does not give any intuitive information about the characteristic

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inner structure of this manifold of paths, e.g. in contrast to the paths of charged particles in an electromagnetic field. The dissatisfaction with this situation is masterly expressed by Einstein [1]: «I quote Galilei's law of inertia as an example (for the mixing of statements about the means of description and statements about the object to be described). It reads in detailed formulation necessarily as follows: Matter points, that are sufficiently separated from each other, move uniformly in a straight line – *provided that the motion is related to a suitably moving coordinate system and that the time is suitably defined.* Who does not feel the painfulness of such a formulation? But omitting the postscript would imply a dishonesty». (Our translation). Even a formally covariant characterization of free fall paths by $T^a\nabla_a T^b = 0$ with $T^a = \frac{dx^a}{d\tau}$ does not contribute very much to a (geometric) comprehension of this path structure. In some, even modern textbooks on classical mechanics the lack of clarification of these basic elements of physics has the consequence that the introduction of «free particles» and the formulation of Galilei's law of inertia come near to a vicious circle.

In order to avoid this danger of circularity, the safest way is to reduce, at least in principle, all statements to the most primitive, experimentally decidable facts, and these seem to be space-time coincidences (events). Historically, the necessity for such a reduction was first stressed in 1915-1916 by Kretschmann [2] and Einstein [3], in the sequel of Einstein's painful wavering between only partly and fully covariant formulations of a relativistic gravitation theory: «Reality is nothing but the totality of space-time point coincidences» [4]. Concerning the free fall paths and Galilei's principle of inertia, this poses the natural question: what is the minimal number of particle paths and points of intersection between them, that allows to characterize uniquely the free fall paths, in distinction from other path structures? In starting this analysis, we have in mind at first the paths of free particles in a 2-dimensional (plane) gravitation-free region, i.e. the globally straight paths of classical mechanics. However, it turns out that this analysis and its results transform in a natural way and nearly unchanged to an n-dimensional space, respectively spacetime, and also to the local characterization of free particles in general relativity.

In order to discriminate between different types of paths (e.g. «straight» and «curved» paths) it is necessary to consider at least 3 points on each of them. In order that these points help to formulate some characteristic (lawful) properties of paths, the points have to be defined not only by the (inevitable) crossing of 2 paths but by the crossing of at least 3 paths. (Duality between paths and points!) The simplest «figure» of this type is easily seen to consist of 7 paths and 7 points, and it is well known in geometry as the simplest example of a finite projective plane, the so-called Fano configuration [5]. However, a realization of such a figure is possible only in an abstract «plane» over a field of characteristic 2, and not in the «physical» continuous and ordered plane. A similar situation occurs for the figure with 8 paths and 8 (nontrivial) points.

The simplest generic «confined configuration», that is realizable in the normal plane consists of 9 paths and 9 points, and is known from ancient times as the Pappus figure (fig. 1): *If the vertices A, B, A', B', A'', B'' of a plane hexagon lie alternately on two lines, then the pairs of opposite sides meet in collinear points e_1, e_2, e_3.* We consider it a most remarkable fact (albeit it is mentioned nowhere in the physics literature) that this mathematically simplest nontrivial coincidence figure allows, at least in the plane, to characterize uniquely the simplest possible paths for physical bodies, the free fall paths, as was first shown by Hilbert [6]: From the purely geometric properties of the Pappus figure together with the parallel axiom, Hilbert developed a coordinatization of the plane (Streckenrechnung) that obeys all laws of the real number field, and in which the basic paths fulfil linear equations (are straight lines). Nevertheless, the Pappus theorem is not sufficient to «construct» the physical space-time, because this theorem is not generalizable in a natural way to more than 2 dimensions.

In order to build up the inertial structure of the 4-dimensional (or higher dimensional) space-time from an incidence structure, the simplest possibility is provided by the second fundamental theorem of projective geometry, the Desargues theorem with 10 paths and 10 (nontrivial) points (fig. 2): *If two triangles (A, A', A'' and B, B', B'') are perspective from a point O, they are perspective from a line (corresponding triangle sides meet in collinear points (e_1, e_2, e_3)).* On the one hand, this theorem (in the plane) is a necessary and sufficient condition for embedding this plane into a higher-dimensional space [6], on the other hand also the Desargues property of a path structure allows, together with the parallel axiom, to develop a coordinatization in which the basic paths are straight [6]. Therefore, the Desargues theorem represents the simplest procedure to characterize in an operative way an inertial frame in 3 and more dimensions on the basis of (free particle) paths and their intersections.

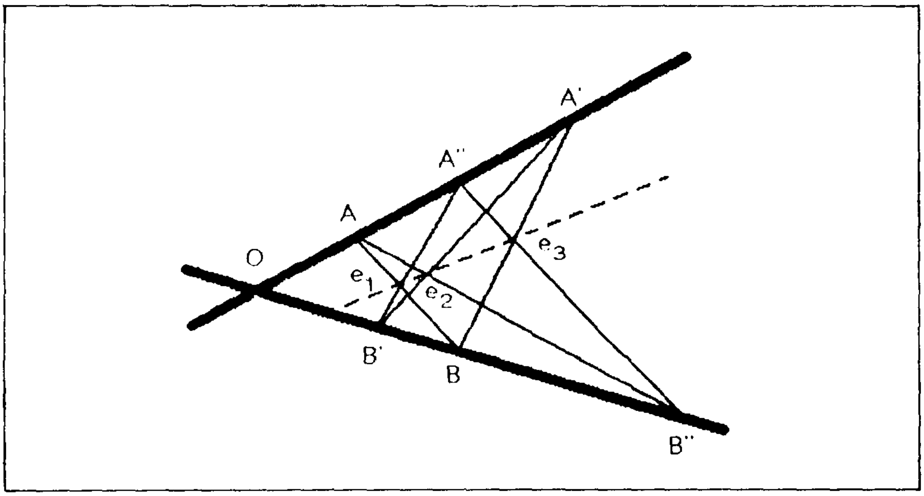

Fig. 1. The Pappus figure for straight lines in the plane.

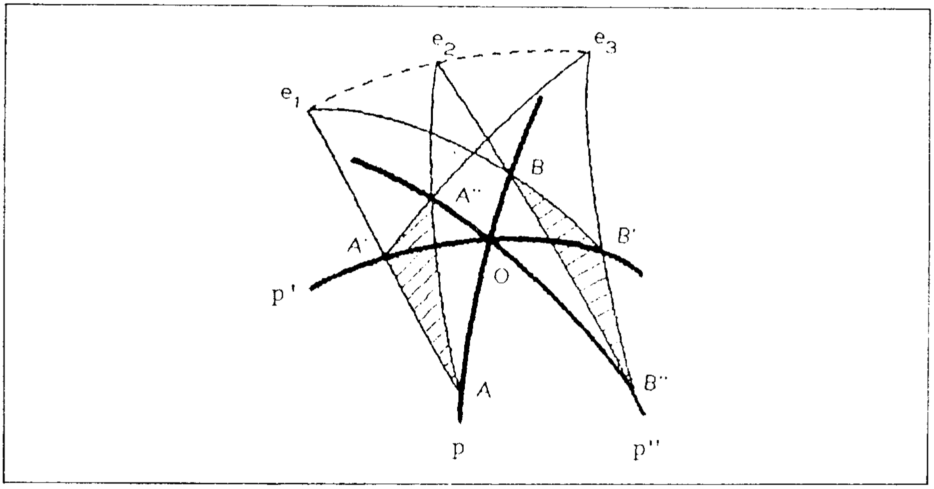

Fig. 2. The Desargues figure, here illustrated for the projective geodesics of general relativity, in two or three dimensions..

The success and the simplicity of this characterization of the inertial structures of classical mechanics and special relativity suggest now to look for a generalization of this procedure to general relativity, a theory that likewise can be built on (particle and light) paths and their intersection points (events) as basic elements. The only global element in the above analysis was the parallel axiom; it has to be and will be discarded in the following analysis of the local theory «general relativity». A «reduction» of the Riemannian geometry of general relativity to the more elementary projective structure (of the free fall paths) and conformal structure (of light paths) goes back to the work of H. Weyl [7]. However, an axiomatic foundation of these structures directly upon the free fall paths and the light rays, without presupposing a metric structure, was not successfully worked out before the seminal paper of Ehlers, Pirani and Schild [8]. Herein, the conformal structure is based solely on incidence properties of light rays with an (arbitrary) particle path, and on topological connectivity properties (of manifolds of light rays). The conformal «metric» $g_{ab}(x^c)$, derived therefrom up to a gauge factor $\Omega^2(x^c)$, can then in principle be constructed locally, in a given coordinate system, by 9 light rays. In contrast, the projective structure was defined in [8] by the standard coordinate-dependent property $\frac{d^2x^a}{d\tau^2} = 0$, and not by incidence properties or other elementary and operative facts about free fall paths.

In subsequent publications by the same authors [9, 10], the goal of a reduction to more elementary properties was mentioned, but no systematic construction of this type was presented. The first purely geometric and intuitive characterization of the free fall paths of general relativity was given by Ehlers and Köhler [11]: The free fall path structure is uniquely singled out among the general ones by admitting at each point approximate symmetries that are induced by a dilatation, or equivalently by a transitive action in the set of projective directions (i.e. by maximal local isotropy). This type of characterization was later also formulated in the language of jet bundles [12], and it was shown [13] that the degree of micro-isotropy of a path structure, that makes it compatible with the conformal structure (according to [8]), is already sufficient to single out the free fall paths. Although these descriptions of the free fall structure by local symmetry arguments are coordinate-independent and geometrically intuitive, and may have their own merits, we should like to argue that a characterization solely by incidence properties as given by a Desargues-like theorem is more elementary and fits more natural into the scheme of a projective structure and to the construction [8] of the conformal light cone structure (for which a characterization solely by symmetry arguments seems not to be available).

In pursuing the goal to characterize the free fall paths of general relativity solely by incidence properties, some years ago the following conjecture was formulated [14]:

THEOREM. *A path structure is a free fall (or inertial) structure if and only if for any point O of the manifold M an ϵ_0-neighbourhood can be found such that the path structure obeys the Desargues property in order ϵ_0^2.* ■

Here, M is an n-dimensional manifold with differential topology; a general path structure is a set of paths (unparametrized, sufficiently smooth curves) in M such that through every point of M and every direction at that point there passes exactly one path. Fulfilment of the Desargues property in order ϵ_0^2 means that the intersection points e_1, e_2, e_3 of figure 2 exist up to corrections of order ϵ_0^3, and that e_3 lies on the path e_1e_2 up to corrections of order ϵ_0^3, for all Desargues configurations which are confined to the ϵ_0-neighbourhood of O.

That the free fall paths (projective geodesics) of general relativity fulfil this local Desargues property, is easily seen: In an arbitrary coordinate system $x^a (a = 1, \dots, n)$ in the ϵ_0-neighbourhood of the origin O, a free fall path through P can be represented by

$$x^a(\tau) = x^a(P) + \tau\, u^a + \frac{1}{2}\tau^2\, \Gamma^a_{bc}(P)\, u^b(P)\, u^c(P) + O(\epsilon_0^3)\,, \tag{1.1}$$

with $u^a(P) := \frac{dx^a}{d\tau}(P)$, and where $x^a(P)$ and the parameter τ are of order ϵ_0. If the local inertial system at O is chosen, the projective connections fulfil $\Gamma^a_{bc}(O) = 0$, and $\Gamma^a_{bc}(P) = O(\epsilon_0)$, if the Γ^a_{bc} are C^1-functions. Together with $\tau = O(\epsilon_0)$, the third term on the right hand side of (1.1) therefore is of order ϵ_0^3, so that the general path in order ϵ_0^2 is given by the linear equation

$$x^a(\tau) = x^a(P) + \tau u^a\,.$$

That such linear paths fulfil the Desargues property, is a standard result of projective geometry [5]. A proof for the reverse fact that the local Desargues property uniquely leads to the free fall paths of general relativity, was initiated in [14], but not completed in a satisfying way. A complete proof will now be presented in the following. In sections 2 and 3, the special case will be considered that the directions of the paths p, p', p'' are linearly dependent in O (see fig. 2). The existence of the intersection points e_1, e_2, e_3 in order ϵ_0^2, due to the Desargues theorem, guarantees that in this case the whole Desargues configuration can be reduced in this order to a 2-dimensional submanifold. (This is the so-called surface-forming property of the projective geodesics that played also a major role in [9] and [10].) As usual in projective geometry, the situation in 2 dimensions is special, and the most difficult to prove. In section 3, first a functional equation for the central «acceleration function» $\vec{B}(\vec{u})$ of the path structure is derived from the Desargues property. Then it is proven, that solutions of this functional equation necessarily depend in a linear or symmetrically bilinear form on the components of $\vec{u}$. However, such dependences can be eliminated ($\vec{B}(\vec{u})$ be made identically zero) by suitable parameter and coordinate transformations. As a side-remark, we should like to point to the noteworthy fact that the simplest acceleration (or force-) function $K^a(u^b)$ that cannot be eliminated in this way, and that obeys the local constancy of the light velocity in the form $K^a u_a = 0$, is of the form $K^a = F^{ab}u_b$ with an antisymmetric F^{ab}, and therefore is realized by the only fundamental long-range force in nature, electromagnetism. In section 4, the proof for the general 3-dimensional respectively n-dimensional case is given, which turns out to consist of a simple and more or less formal extension of the 2-dimensional results.

In connection with the central role of the Desargues theorem for the foundations of classical mechanics, special relativity, and general relativity, that is the main objective of this paper, we should like to add two more remarks: A nice example for the successful application of the Desargues theorem (and the only one known to us from the physics literature) is the construction of a «geometrodynamic clock» (solely with the basic elements of the projective and conformal structure) by Castagnino [15], that considerably simplifies the original proposal by Marzke and Wheeler [16]. Finally, it seems noteworthy that the Desargues theorem supplies a simple construction of spacelike projective geodesics solely out of timelike ones, i.e. out of real free fall paths (and without the help of light rays, as is otherwise usual in the Einstein synchronization procedure): Already in a 2-dimensional Minkowski diagram it easily can be seen that it is possible to choose the paths p, p', p'' and all the triangle sides of figure 2 timelike, but the path $e_1e_2e_3$ spacelike, and to construct (in principle) all points of the path $e_1e_2e_3$ by suitably varying the triangles.

2. PREPARATIONS

2.1 Coordinate representation of the path structure

In order to shorten the equations and to simplify the proof, in this paper only the 3-dimensional case of Desargues' construction will be carried out in detail. Some remarks for the case of an n-dimensional manifold will be given in section 4.2. Because we may carry out Desargues' construction in the whole ϵ_0-neighbourhood of O, we can carry it out in any ϵ-neighbourhood with $0 < \epsilon \leq \epsilon_0$. In order to carry out Desargues' construction, we have to intersect paths in an ϵ_0-neighbourhood. So we have to start with the equation of such a path. With respect to any chart, the second degree Taylor expansion of a path at a point $\vec{x}_0$ is

$$\vec{x}(\tau) = \vec{x}_0 + \tau\frac{d\vec{x}}{d\tau}(0) + \frac{1}{2}\tau^2\frac{d^2\vec{x}}{d\tau^2}(0) + \frac{1}{6}\tau^3\frac{d^3\vec{x}}{d\tau^3}(\Theta\tau)$$

with $\Theta \in [0,1]$, and $\vec{x}$ representing a general 3-dimensional vector. Because a path is determined by a point and the direction at this point, there exists an «acceleration field» $\vec{A}(\vec{x}_0, \vec{u})$ such that

$$\frac{1}{2}\frac{d^2\vec{x}}{d\tau^2}(0) =: \vec{A}\left(\vec{x}_0, \frac{d\vec{x}}{d\tau}(0)\right) .$$

This acceleration field is not uniquely defined, because another parameter choice $\sigma(\tau)$ produces another acceleration field. In the following, one special representative field will be used. Because the path depends only on the direction at a point, and not on $|\vec{u}|$, the acceleration field has the property

$$\vec{A}(\vec{x}, \lambda\vec{u}) = \lambda^2\vec{A}(\vec{x},\vec{u}) \quad \text{for all } \lambda \neq 0 .$$

A second degree Taylor expansion only makes sense if

$$\frac{d^3\vec{x}}{d\tau^3}(\Theta\tau) = 2\frac{d}{d\tau}\left(\vec{A}\left(\vec{x}(\tau), \frac{d\vec{x}}{d\tau}(\tau)\right)\right)(\Theta\tau)$$

exists. Therefore it is appropriate to require that $\partial_{\vec{x}}\vec{A}$ and $\partial_{\vec{u}}\vec{A}$ exist.

Now we choose a preliminary chart by taking three paths of the path structure, that intersect at a point O, and whose tangential vectors are linearly independent at O. We take these paths as axes of a chart. Because the axes are given by the equations

$$\vec{x}_i(\tau) = \tau\vec{u}_i + \tau^2\vec{A}(\vec{0},\vec{u}_i) + O(\tau^3) \quad i = 1,2,3$$

with $\vec{u}_1 = (u^1,0,0), \vec{u}_2 = (0,u^2,0), \vec{u}_3 = (0,0,u^3)$, the acceleration field has the property $A^i(\vec{0},\vec{u}_j) = 0$ for $i \neq j$. The scale on each axis can be chosen such that

$$\vec{A}(\vec{0},\vec{u}_i) = 0 \quad \text{for all} \quad i = 1,2,3.$$

Take now two points P, Q in an ϵ-neighbourhood of O. Then there exists one unique path, connecting P and Q. The equation of this path will now be derived. The equation of a path, starting at P with velocity $\vec{u}$ is

$$\vec{x}(\epsilon\rho) = \vec{x}_P + \epsilon\rho\vec{u} + \epsilon^2\rho^2\vec{A}(\vec{x}_P, \vec{u}) + O(\epsilon^3)\ .$$

Because P lies in the ϵ-neighbourhood of O it follows

$$\vec{A}(\vec{x}_P, \vec{u}) = \vec{A}(\vec{0}, \vec{u}) + O(\epsilon).$$

With $\vec{B}(\vec{u}) := \vec{A}(\vec{0}, \vec{u})$ we get

$$\vec{x}(\epsilon\rho) = \vec{x}_P + \epsilon\rho\vec{u} + \epsilon^2\rho^2\vec{B}(\vec{u}) + O(\epsilon^3)\ .$$

By demanding $\vec{x}(\epsilon) = \vec{x}_Q$ we get the equation

$$\vec{x}_Q - \vec{x}_P = \epsilon\vec{u} + \epsilon^2\vec{B}(\vec{u}) + O(\epsilon^3)\ .$$

In order to solve this equation for $\vec{u}$, take $\vec{u} = \vec{u}_0 + \epsilon\vec{u}_1 + \dots$. Then we get

$$\vec{u}_0 = \frac{\vec{x}_Q - \vec{x}_P}{\epsilon},\ \vec{u}_1 = -\vec{B}(\vec{u}_0)\ .$$

Therefore, to second order, the equation of a path through P and Q has the form

$$\vec{x}(\epsilon\rho) = \vec{x}_P + \rho\vec{v} + \epsilon^2\rho(\rho - 1)\ \vec{B}\left(\frac{\vec{v}}{\epsilon}\right) + O(\epsilon^3) \tag{2.1}$$

with $\vec{v} := \vec{x}_Q - \vec{x}_P$, and ρ restricted such that the curve doesn' t leave the ϵ-neighbourhood of O.

$\vec{B}(\vec{u})$ has the following properties:

$$\vec{B}(u, 0, 0) = \vec{B}(0, u, 0) = \vec{B}(0, 0, u) = 0 \quad \text{for all} \quad u \in \mathbf{R}\ . \tag{2.2}$$

$$\vec{B}(\lambda\vec{u}) = \lambda^2\vec{B}(\vec{u})\ . \tag{2.3}$$

(2.4) $\vec{B}$ is continuous.

(2.5) $\partial_{\vec{u}}\vec{B}(\vec{u})$ exists for all $\vec{u} \in \mathbf{R}^3$.

In order to describe Desargues' construction, it is necessary to construct the intersection of two general paths (if such an intersection exists at all). Given two paths

$$\vec{x}(\epsilon\rho) = \vec{x}_P + \rho\vec{v} + \epsilon^2\rho(\rho - 1)\vec{B}(\frac{\vec{v}}{\epsilon}) + O(\epsilon^3)\ ,$$

$$\vec{x}(\epsilon\sigma) = \vec{x}_{P'} + \sigma\vec{v}' + \epsilon^2\sigma(\sigma - 1)\vec{B}(\frac{\vec{v}'}{\epsilon}) + O(\epsilon^3)\ ,$$

intersection means

$$\vec{x}_P + \rho\vec{v} + \epsilon^2\rho(\rho - 1)\vec{B}\left(\frac{\vec{v}}{\epsilon}\right)$$
$$= \vec{x}_{P'} + \sigma\vec{v'} + \epsilon^2\sigma(\sigma - 1)\vec{B}\left(\frac{\vec{v'}}{\epsilon}\right) + O(\epsilon^3)\ .$$

In order to solve this equation, expand ρ and σ with respect to ϵ

$$\rho = \rho_0 + \epsilon\rho_1 + \ldots, \sigma = \sigma_0 + \epsilon\sigma_1 + \ldots\ .$$

Then, the terms of order ϵ give

$$\vec{x}_P + \rho_0\vec{v} = \vec{x}_{P'} + \sigma_0\vec{v'}, \tag{2.6}$$

and the terms of order ϵ^2 give

$$\rho_1\vec{v} - \sigma_1\vec{v'} = -\rho_0(\rho_0 - 1)\vec{B}(\frac{\vec{v}}{\epsilon}) + \sigma_0(\sigma_0 - 1)\vec{B}(\frac{\vec{v'}}{\epsilon})\ . \tag{2.7}$$

Obviously equation (2.6) cannot be solved for all $\vec{x}_P, \vec{x}_{P'}, \vec{v}, \vec{v'} \in \mathbf{R}^3$. In order that equation (2.7) for ρ_1 and σ_1 can be solved, the function $\vec{B}$ has to fulfil appropriate conditions to which we come back in section 2.2. If these equations can be solved, the point of intersection of order ϵ^2 is

$$\vec{x}_{\text{inters}} = \vec{x}_P + (\rho_0 + \epsilon\rho_1)\vec{v} + \epsilon^2\rho_0(\rho_0 - 1)\vec{B}\left(\frac{\vec{v}}{\epsilon}\right) + O(\epsilon^3)\ . \tag{2.8}$$

2.2 Construction of a 2-dimensional subspace

If we take two paths p and p' , which intersect at one point O , Desargues' construction can be used to define a surface. This surface is defined by the following property: any point of the surface can be reached by a path connecting a point on the path p and another point on the path p' . Then Desargues property requires that this surface really is a 2-dimensional subspace.

If the Desargues property holds for a given path structure, we can choose in particular p as x^1-axis , and p' as x^2-axis . On these paths we choose the points

$$A = \epsilon(1, 0, 0), \quad B = \epsilon(-1, 0, 0)$$
$$A' = \epsilon(0, a, 0), \quad B' = \epsilon(0, 1, 0)$$

with $a \in \mathbb{R}$ and $0 < \epsilon \leq \epsilon_0$. The intersection point of AA' and $B'B$ must exist. According to equation (2.6) we have then in order ϵ

$$\begin{pmatrix} 1-\rho_0 \\ a\rho_0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\sigma_0 \\ 1-\sigma_0 \\ 0 \end{pmatrix} .$$

Therefore we get

$$\rho_0 = \frac{2}{1+a} \text{ and } \sigma_0 = \frac{1-a}{1+a} .$$

According to equation (2.7) we have in order ϵ^2

$$\begin{pmatrix} -1 \\ a \\ 0 \end{pmatrix} \rho_1 + \rho_0(\rho_0 - 1)\vec{B}(-1, a, 0) =$$

$$\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \sigma_1 + \sigma_0(\sigma_0 - 1)\vec{B}(-1, -1, 0) .$$

The third component of this equation yields

$$B^3(-1, a, 0) = -aB^3(-1, -1, 0) = -a \text{ const} .$$

With equation (2.3) we get

$$B^3(u^1, u^2, 0) = \Gamma^3_{12}\, u^1 u^2 \quad \text{with } \Gamma^3_{12} = \text{const} . \tag{2.9}$$

Herewith it can be shown that the coordinates of a point $\vec{x}$ that lies in the surface $P_{1,2}$, which is defined by the x^1-axis and the x^2-axis, have the form

$$\vec{x} = (x^1, x^2, \Gamma^3_{12}\, x^1 x^2) . \tag{2.10}$$

We can use the properties (2.9) and (2.10) to simplify the equation of a path connecting two points of order ϵ that lie in the surface $P_{1,2}$. To this end we change the chart (in a way that leaves invariant the axes).

$$x'^1 = x^1 , \quad x'^2 = x^2 ,$$

$$x'^3 = x^3 - \Gamma^3_{12} x^1 x^2 .$$

In this chart, the equation of a path through two points P, Q (order ϵ) of the surface $P_{1,2}$ is

$$\vec{x}'(\epsilon\rho) = \vec{x}'_P + \rho\vec{v}' + \epsilon^2\rho(\rho-1)\begin{pmatrix} B^1(\frac{v'^1}{\epsilon}, \frac{v'^2}{\epsilon}, 0) \\ B^2(\frac{v'^1}{\epsilon}, \frac{v'^2}{\epsilon}, 0) \\ 0 \end{pmatrix} + O(\epsilon^3) \tag{2.11}$$

with $\vec{v}' := \vec{x}'_Q - \vec{x}'_P$.

Proof. The equation of a path through the two points P and Q is

$$\vec{x}(\epsilon\rho) = \vec{x}_P + \rho\vec{v} + \epsilon^2\rho(\rho-1)\vec{B}\left(\frac{\vec{v}}{\epsilon}\right) + O(\epsilon^3) \tag{2.12}$$

with $\vec{v} = \vec{x}_Q - \vec{x}_P$. Because $\vec{v} = (v^1, v^2, 0) + O(\epsilon^2)$ and $\vec{B}$ is differentiable we get

$$\vec{B}\left(\frac{\vec{v}}{\epsilon}\right) = \vec{B}\left(\frac{(v^1, v^2, 0)}{\epsilon} + O(\epsilon)\right) = \vec{B}\left(\frac{v^1}{\epsilon}, \frac{v^2}{\epsilon}, 0\right) + O(\epsilon)\ .$$

If we insert this result into equation (2.12) it follows

$$\begin{aligned} \vec{x}(\epsilon\rho) &= \vec{x}_P + \rho\vec{v} + \epsilon^2\rho(\rho-1)\vec{B}\left(\frac{v^1}{\epsilon}, \frac{v^2}{\epsilon}, 0\right) + O(\epsilon^3) \\ &= \vec{x}_P + \rho\vec{v} + \epsilon^2\rho(\rho-1)\left(B^1, B^2, \Gamma^3_{12}\frac{v^1v^2}{\epsilon^2}\right) + O(\epsilon^3)\ . \end{aligned}$$

Now we have to transform this equation.

$$\begin{aligned} x'^1(\epsilon\rho) &= x^1(\epsilon\rho) \\ x'^2(\epsilon\rho) &= x^2(\epsilon\rho) \\ x'^3(\epsilon\rho) &= x^3(\epsilon\rho) - \Gamma^3_{12}x^1(\epsilon\rho)x^2(\epsilon\rho) \\ &= x^3_P + \rho(x^3_Q - x^3_P) + \rho(\rho-1)\Gamma^3_{12} \\ &\quad (x^1_Q - x^1_P)(x^2_Q - x^2_P) \\ &\quad - \Gamma^3_{12}[x^1_P + \rho(x^1_Q - x^1_P) + O(\epsilon^2)] \\ &\quad [x^2_P + \rho(x^2_Q - x^2_P) + O(\epsilon^2)] + O(\epsilon^3) \\ &= x^3_P - \Gamma^3_{12}x^1_Px^2_P \\ &\quad + \rho[(x^3_Q - \Gamma^3_{12}\, x^1_Qx^2_Q) - (x^3_P - \Gamma^3_{12}x^1_Px^2_P)] + O(\epsilon^3) \\ &= x'^3_P + \rho[x'^3_Q - x'^3_P] + O(\epsilon^3)\ . \end{aligned}$$

Because we use only the new chart in the following, the coordinates of it will again be denoted by x^i.

3. PROOF IN 2 DIMENSIONS

3.1 Derivation of a functional equation

In this chapter we just regard points P, Q with $x^3_Q = x^3_P = 0$, and paths through these points. Because the third component of such a path is zero according to equation (2.11), we can drop the third coordinate in this section. A further simplification of this equation can be achieved by a suitable choice of the parameter. If $v^2 \neq 0$, choose

$$\rho = \sigma - \epsilon^2 \sigma(\sigma - 1) \frac{B^2\left(\frac{v^1}{\epsilon}, \frac{v^2}{\epsilon}, 0\right)}{v^2} .$$

This yields

$$\vec{x}(\epsilon\rho) = \begin{pmatrix} x^1_P \\ x^2_P \end{pmatrix} + \sigma \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} + \epsilon^2 \sigma(\sigma - 1) \begin{pmatrix} B^1 - \frac{v^1}{v^2} B^2 \\ 0 \end{pmatrix} + O(\epsilon^3)$$

with $\vec{v} = \vec{x}_Q - \vec{x}_P$. Define now

$$b(u^1, u^2) := \begin{cases} B^1(u^1, u^2, 0) - \frac{u^1}{u^2} B^2(u^1, u^2, 0) & ; u^2 \neq 0 \\ -u^1 \partial_{u^2} B^2(u^1, 0, 0) & ; u^2 = 0 . \end{cases}$$

According to the properties (2.2) ... (2.5) of $\vec{B}$, b is continuous for all $\vec{u}$ and differentiable for $u^2 \neq 0$. Then the equation of a path is

$$\vec{x}(\epsilon\rho) = \begin{pmatrix} x^1_P \\ x^2_P \end{pmatrix} + \sigma \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} + \epsilon^2 \sigma(\sigma - 1) \begin{pmatrix} b\left(\frac{v^1}{\epsilon}, \frac{v^2}{\epsilon}\right) \\ 0 \end{pmatrix} + O(\epsilon^3) .$$

We see, that this equation also holds for $v^2 = 0$. Because of $\vec{B}(\lambda\vec{u}) = \lambda^2 \vec{B}(\vec{u})$, $b(u^1, u^2)$ fulfils the following equation:

$$b(\lambda u^1, \lambda u^2) = \lambda^2 b(u^1, u^2) .$$

Define:

$$g(x) := b(1, x) .$$

Then

1) $\quad (v^1)^2 \, g\left(\frac{v^2}{v^1}\right) = \epsilon^2 b\left(\frac{v^1}{\epsilon}, \frac{v^2}{\epsilon}\right) \quad (\text{for } v^1 \neq 0) .$

2) $\quad g$ is continuous.

3) $\quad \frac{dg}{dx}(x)$ exists (at least) for $x \neq 0$.

Now, the final equation of a path through the points P, Q of order ϵ is in second order

$$\vec{x}(\epsilon\rho) = \begin{cases} \vec{x}_P + \rho\vec{v} + \rho(\rho-1)\,(v^1)^2 \begin{pmatrix} g\left(\frac{v^2}{v^1}\right) \\ 0 \end{pmatrix} + O(\epsilon^3) & ;v^1 \neq 0 \\ \vec{x}_P + \rho\vec{v} + O(\epsilon^3) & ;v^1 = 0 \end{cases}$$

$$(\vec{v} = \vec{x}_Q - \vec{x}_P) .$$

In the following we carry out Desargues' construction, in order to determine the function g.

It turns out to be sufficient to consider the following special case:

$$\begin{array}{lll} p : x^1\text{-axis}; & A = \epsilon(1,0); & B = \epsilon(-1,0) \\ p' : x^2\text{-axis}; & A' = \epsilon(0,a); & B' = \epsilon(0,1) \\ p'' : \text{ path } OA'' : & A'' = \epsilon(1,1); & B'' \text{anywhere on } OA'' . \end{array}$$

Because B'' lies anywhere on OA'' it has the coordinates

$$B'' = \epsilon(c,c) + \epsilon^2 c(c-1)(g(1),0) + O(\epsilon^3), \quad c \in \mathbf{R}.$$

Determination of e_1 by intersection of AA' and BB':

$$AA' : \vec{x}(\epsilon\rho) = \epsilon \begin{pmatrix} 1-\rho \\ a\rho \end{pmatrix} + \epsilon^2 \rho(\rho-1) \begin{pmatrix} g(-a) \\ 0 \end{pmatrix} + O(\epsilon^3) .$$

$$B'B : \vec{x}(\epsilon\sigma) = \epsilon \begin{pmatrix} -\sigma \\ 1-\sigma \end{pmatrix} + \epsilon^2 \sigma(\sigma-1) + \begin{pmatrix} g(1) \\ 0 \end{pmatrix} + O(\epsilon^3) .$$

Intersection yields

$$\rho = \frac{2}{1+a} + \epsilon \frac{2(1-a)}{(1+a)^3}(g(-a) + ag(1)) + \dots,$$

(3.1)
$$\sigma = \frac{1-a}{1+a} - \epsilon \frac{2a(1-a)}{(1+a)^3}(g(-a) + ag(1)) + \dots,$$

$$\vec{e}_1 = \epsilon \begin{pmatrix} \frac{a-1}{1+a} \\ \frac{2a}{1+a} \end{pmatrix} + \epsilon^2 \frac{2a(1-a)}{(1+a)^3} \begin{pmatrix} g(-a) - g(1) \\ g(-a) + ag(1) \end{pmatrix} + O(\epsilon^3) .$$

Determination of e_2 by intersection of AA'' and BB'':

$$AA'' : \vec{x}(\epsilon\rho) = \epsilon \begin{pmatrix} 1 \\ \rho \end{pmatrix} + O(\epsilon^3) .$$

$$\begin{aligned} BB'' : \vec{x}(\epsilon\sigma) = & \epsilon \begin{pmatrix} -1 + (1+c)\sigma \\ c\sigma \end{pmatrix} + \epsilon^2 c(c-1)\sigma \begin{pmatrix} g(1) \\ 0 \end{pmatrix} \\ & + \epsilon^2 \sigma(\sigma - 1)[1 + c + \epsilon g(1) c(c-1)]^2 \cdot \\ & \cdot \begin{pmatrix} g\left(\frac{c}{1+c+O(\epsilon)}\right) \\ 0 \end{pmatrix} + O(\epsilon^3) . \end{aligned}$$

Because of $g(x + \epsilon h) = g(x) + O(\epsilon)$ (g is differentiable), it follows

$$\begin{aligned} BB'' : \vec{x}(\epsilon\sigma) = & \epsilon \begin{pmatrix} -1 + (1+c)\sigma \\ c\sigma \end{pmatrix} + \epsilon^2 \sigma c(c-1) \begin{pmatrix} g(1) \\ 0 \end{pmatrix} \\ & + \epsilon^2 \sigma(\sigma - 1)[1 + c]^2 \begin{pmatrix} g\left(\frac{c}{1+c}\right) \\ 0 \end{pmatrix} + O(\epsilon^3) . \end{aligned}$$

Intersection yields

$$\sigma = \frac{2}{1+c} + \frac{2(1-c)}{(1+c)^2} \left(cg(1) - (1+c) g\left(\frac{c}{1+c}\right)\right) \epsilon + \dots,$$

$$\rho = c\sigma$$

$$\begin{aligned} \vec{e}_2 = & \epsilon \begin{pmatrix} 1 \\ \frac{2c}{1+c} \end{pmatrix} + \\ & \epsilon^2 \frac{2c(1-c)}{(1+c)^3} \begin{pmatrix} 0 \\ c(1+c) g(1) - (1+c)^2 g\left(\frac{c}{1+c}\right) \end{pmatrix} + O(\epsilon^3) . \end{aligned} \tag{3.2}$$

Determination of e_3 by intersection of $A'A''$ and $B'B''$:

$$\begin{aligned} A'A'' : \vec{x}(\epsilon\rho) = & \epsilon \begin{pmatrix} \rho \\ a + (1-a)\rho \end{pmatrix} + \epsilon^2 \rho(\rho - 1) \\ & \begin{pmatrix} g(1-a) \\ 0 \end{pmatrix} + O(\epsilon^3) . \end{aligned}$$

$$\begin{aligned} B'B'' : \vec{x}(\epsilon\sigma) = & \epsilon \begin{pmatrix} c\sigma \\ 1 + (c-1)\sigma \end{pmatrix} + \epsilon^2 \sigma c(c-1) \begin{pmatrix} g(1) \\ 0 \end{pmatrix} \\ & + \epsilon^2 \sigma(\sigma - 1) c^2 \begin{pmatrix} g\left(\frac{c-1}{c}\right) \\ 0 \end{pmatrix} + O(\epsilon^3) . \end{aligned}$$

Intersection yields

$$
\begin{aligned}
\rho &= \frac{c(1-a)}{1-ac} + \epsilon \frac{c(1-a)(c-1)^2}{(1-ac)^3} \cdot \\
&\quad \left(g(1-a) - (1-ac)g(1) - acg\left(\frac{c-1}{c}\right)\right) + \dots, \\
\sigma &= \frac{1-a}{1-ac} + \epsilon \frac{c(1-a)^2(c-1)}{(1-ac)^3} \cdot \\
&\quad \left(g(1-a) - (1-ac)g(1) - acg\left(\frac{c-1}{c}\right)\right) + \dots, \\
\vec{e}_3 &= \frac{\epsilon}{1-ac} \begin{pmatrix} c(1-a) \\ a+c-2ac \end{pmatrix} + \epsilon^2 \frac{c(1-a)^2(c-1)^2}{(1-ac)^3} \cdot \\
&\quad \begin{pmatrix} \frac{c}{c-1} g(1-a) - \frac{ac}{1-a} g\left(\frac{c-1}{c}\right) - \frac{1-ac}{1-a} g(1) \\ g(1-a) - acg\left(\frac{c-1}{c}\right) - (1-ac)g(1) \end{pmatrix} + O(\epsilon^3) .
\end{aligned}
\tag{3.3}
$$

Application of Desargues' condition:

Desargues requires, that e_3 lies on the path through e_1 and e_2. The path through e_1 and e_2 has the equation

$$
\begin{aligned}
\vec{x}(\epsilon\rho) =& \epsilon(1-\rho) \begin{pmatrix} -\frac{1-a}{1+a} \\ \frac{2a}{1+a} \end{pmatrix} + \epsilon\rho \begin{pmatrix} 1 \\ \frac{2c}{1+c} \end{pmatrix} \\
&+ \epsilon^2 (1-\rho) \frac{2a(1-a)}{(1+a)^3} \begin{pmatrix} g(-a) - g(1) \\ g(-a) + ag(1) \end{pmatrix} \\
&- \epsilon^2 \rho \frac{2c(c-1)}{(1+c)^2} \begin{pmatrix} 0 \\ cg(1) - (1+c)g\left(\frac{c}{1+c}\right) \end{pmatrix} \\
&+ \epsilon^2 \rho(\rho-1) \frac{4}{(1+a)^2} \begin{pmatrix} g\left(\frac{c-a}{1+c}\right) \\ 0 \end{pmatrix} + O(\epsilon^3) .
\end{aligned}
$$

The condition is, that there exists a $\rho = \rho_0 + \epsilon\rho_1 + \dots$ such that

$$
\vec{e}_3 = \vec{x}(\epsilon\rho_0 + \epsilon^2\rho_1 + \dots).
$$

This condition can be separated into order ϵ and order ϵ^2 terms.

The equation of order ϵ is

$$
\begin{pmatrix} \frac{a-1}{1+a} \\ \frac{2a}{1+a} \end{pmatrix} (1-\rho_0) + \begin{pmatrix} 1 \\ \frac{2c}{1+c} \end{pmatrix} \rho_0 = \begin{pmatrix} c\frac{1-a}{1-ac} \\ \frac{a+c-2ac}{1-ac} \end{pmatrix} ,
$$

and has the solution

$$\rho_0 = \frac{(1+c)(1-a)}{2(1-ac)} .$$

The equation of order ϵ^2 is

$$\begin{aligned}
&\begin{pmatrix} -\frac{1-a}{1+a} \\ \frac{2a}{1+a} \end{pmatrix}(-\rho_1) + \begin{pmatrix} 1 \\ \frac{2c}{1+c} \end{pmatrix}\rho_1 \\
&+\frac{2a(1-a)}{(1+a)^3}\,\frac{-(a+1)(c-1)}{2(1-ac)}\begin{pmatrix} g(-a)-g(1) \\ g(-a)+ag(1) \end{pmatrix} \\
&+\frac{2c(1-c)}{(1+c)^2}\,\frac{(1+c)(1-a)}{2(1-ac)}\begin{pmatrix} 0 \\ cg(1)-(1+c)g\left(\frac{c}{1+c}\right) \end{pmatrix} \\
&+\frac{4}{(1+a)^2}\,\frac{(1+c)(a+1)(1-a)(c-1)}{4(1-ac)^2}\begin{pmatrix} g\left(\frac{c-a}{1+c}\right) \\ 0 \end{pmatrix} \\
&=\frac{c(1-a)^2(c-1)^2}{(1-ac)^3}\begin{pmatrix} \frac{c}{c-1}g(1-a)-\frac{ac}{1-a}g(\frac{c-1}{c})-\frac{1-ac}{1-a}g(1) \\ g(1-a)-acg(\frac{c-1}{c})-(1-ac)g(1) \end{pmatrix} .
\end{aligned}$$

These are two equations for one unknown ρ_1 . So, a solution only exists if these equations are linearly dependent. This yields the following condition:

$$\begin{aligned}
&c(1-a^2)g(1-a)+ac^2(c-1)(1+a)g\left(\frac{c-1}{c}\right) \\
&-a(1-ac)g(-a) \\
&+(1-ac)(1+a)c(1+c)g\left(\frac{c}{1+c}\right)-(c-a)(1+c)g(\frac{c-a}{1+c}) \\
&=c(1+2a)(1-ac)g(1) .
\end{aligned} \tag{3.4}$$

This equation does not hold for all $a, c \in \mathbf{R}$. a, c must fulfil the following conditions: First of all, $g(x)$ might not be differentiable at $x = 0$. In deriving the equation of a path through two points, we used the property $g(x+\epsilon h) = g(x) + O(\epsilon)$ in some places. So we have to guarantee, that in these cases $x \neq 0$. This yields the restrictions $c \neq 0, c \neq 1$ and $c \neq a$.

Other restrictions result from the fact, that the points $A, \dots B''$ must be chosen such that the intersection points e_1, e_2 and e_3 lie in the ϵ_0-neighbourhood .

According to equations (3.1), (3.2) and (3.3) these points have the form

$$\vec{e}_i = \epsilon\frac{\vec{a}_i}{1-\xi} + \epsilon^2\frac{\vec{b}_i}{(1-\xi)^3} + O(\epsilon^3), i = 1, 2, 3$$

(For example $\xi = ac$ for $\vec{e}_3$). $\vec{a}_i$ and $\vec{b}_i$ depend on a and c, but are finite vectors. In some expressions $\vec{b}_i$ there appears g in the form $\mu^2 g(\frac{\lambda}{\mu})$, which is not defined at $\mu = 0$. But

$$\lim_{\mu\to 0}\mu^2 g(\frac{\lambda}{\mu}) = \lim_{\mu\to 0} b(\mu,\lambda) = b(0,\lambda) = 0.$$

Now we have to look for the restrictions on ξ. It is clear, that ξ must not be 1. If ξ is near to 1, that means $\xi = 1-\delta$ (without any restrictions $|\delta| \le \epsilon_0 < 1$), the expression $\frac{1}{1-\xi} = \frac{1}{\delta}$ may increase in such a way, that $\vec{e}_i$ leaves the ϵ_0-neighbourhood. This can be avoided by a suitable choice of $\epsilon \le \epsilon_0$: Choose $\epsilon = \delta^2$, then we get

$$\vec{e}_i = \delta\vec{a}_i + \delta b_i .$$

Choosing $\epsilon = \delta^2$ means the following: One takes an ϵ-neighbourhood within the ϵ_0-neighbourhood. In that ϵ-neighbourhood one carries out Desargues' construction. Because the paths are nearly parallel, the intersection points leave the ϵ-neighbourhood, but they remain within the ϵ_0-neighbourhod.

We see therefore, that equation (3.4) holds for all $a, c \in \mathbb{R}$, except $c = 0, c = 1, c = a, a = -1, c = -1$ and $c = \frac{1}{a}$.

3.2 Solution of the functional equation

It is easy to see that $g(x) = g(1) = \text{const}$ fulfils equation (3.4). Now define:

$$f(1-x) := g(x) - g(1).$$

Then f is continuous and differentiable for all $x \neq 1$. Equation (3.4) transforms to

$$\begin{aligned} 0 =& c(1-a^2)f(a) + ac^2(1+a)(c-1)f\left(\frac{1}{c}\right) \\ &- a(1-ac)f(1+a) - (c-a)(1+c)f\left(\frac{1+a}{1+c}\right) \\ &+ (1-ac)(1+a)c(1+c)f\left(\frac{1}{1+c}\right). \end{aligned} \tag{3.5}$$

This equation holds for all $a, c \in \mathbb{R}$ except $a = -1, c = -1, a = \frac{1}{c}, a = c, c = 0$ and $c = 1$. But these restrictions can be dropped because f is continuous.

$f(x)$ fulfils $f(x) = x^2 f(\frac{1}{x})$.

Proof: $a = c$ in equation (3.5) yields

$$k(a) := f(a) - a^2 f(\frac{1}{a}) = f(1+a) - (1+a)^2 f\left(\frac{1}{1+a}\right).$$

k has the following properties:

(3.6) $k(1 + a) = k(a)$.

(3.7) $k(1) = 0$. Therefore k is continuous, because $\lim_{a\to 0} k(a) = \lim_{a\to 0} k(1 + a) = k(1) = 0$.

(3.8) There exists $M > 0$ with $|k(a)| \leq M$ for all $a \in \mathbb{R}$.

(3.9) $k(a) = -a^2 k\left(\frac{1}{a}\right)$.

Insertion of $a = 1$ into equation (3.5) leads to

$$c^2 f\left(\frac{1}{c}\right) - c(1 + c) f\left(\frac{1}{1 + c}\right) = -\frac{1}{2} f(2) + \frac{1 + c}{2} f\left(\frac{2}{1 + c}\right).$$

Therefore

$$-a^2 f\left(\frac{1}{a}\right) + a(1 + a) f\left(\frac{1}{1 + a}\right) = \frac{1}{2} f(2) - \frac{1 + a}{2} f\left(\frac{2}{1 + a}\right). \tag{3.10}$$

Insertion of $c = 1$ into equation (3.5) leads to

$$f(a) - \frac{a}{1 + a} f(1 + a) = \frac{2}{1 + a} f\left(\frac{1 + a}{2}\right) - 2 f\left(\frac{1}{2}\right). \tag{3.11}$$

Add equation (3.10) and equation (3.11)

$$k(a) - \frac{a}{1 + a} k(1 + a) = \frac{1}{2} k(2) + \frac{2}{1 + a} k\left(\frac{1 + a}{2}\right).$$

With properties (3.6) and (3.7) one gets

$$k(a) = 2 k\left(\frac{1 + a}{2}\right).$$

Therefore it holds

$$k(a) = k(1 + a) = 2\ k\left(\frac{1 + (1 + a)}{2}\right) = 2\ k\left(1 + \frac{a}{2}\right) = 2\ k\left(\frac{a}{2}\right)$$

If one applies this equation n times one gets with property (3.9)

$$k(a) = 2^n\ k\left(\frac{a}{2^n}\right) = -2^n\left(\frac{a}{2^n}\right)^2\ k\left(\frac{2^n}{a}\right).$$

This leads to

$$0 \leq |k(a)| = \left| \lim_{n\to\infty} \frac{a^2}{2^n} k(\frac{2^n}{a}) \right| \leq M \cdot \lim_{n\to\infty} \frac{a^2}{2^n} = 0\,.$$

∎

Therefore it holds $k(a) = 0$ and $f(a) = a^2\, f(\frac{1}{a})$.

Using this property and once more the fundamental Desargues functional equation (3.5), it is possible to prove $f(x) = \alpha x$ with $\alpha = \text{const}$.

Define:

$$\tilde{F}(1-x) := \begin{cases} \frac{f(x)}{x} & \text{for } x \neq 0 \\ \frac{df}{dx}(0) & \text{for } x = 0 \end{cases}$$

$$F(x) := \tilde{F}(x) - \tilde{F}(2)$$

Then F has the properties

(3.12) $F(1-x) = F\left(1 - \frac{1}{x}\right);\ F(u) = F\left(\frac{u}{u-1}\right)$.

(3.13) F is continuous.

(3.14) $F(2) = 0$.

With $f(x) = x(F(1-x) + \tilde{F}(2))$ and $f(x) = x^2 f\left(\frac{1}{x}\right)$, equation (3.5) reads

$$(c-a)F\left(\frac{c-a}{1+c}\right) = ac(1-a)F(1-a) + ac(c-1)F(1-c) - a(1-ac)F(-a) + c(1-ac)F(-c)\ . \tag{3.15}$$

Because F is continuous, this equation holds for all $a, c \in \mathbf{R}$.

For $c = 1$ one gets

$$a(F(1-a) - F(-a)) = F\left(\frac{1-a}{2}\right) - F(-1).$$

Insertion of $a = -1$ into this equation yields $F(-1) = F(2) = 0$, and therefore

$$c(F(1-c) - F(-c)) = F\left(\frac{1-c}{2}\right) = F\left(\frac{c-1}{1+c}\right)\ . \tag{3.16}$$

because of property (3.12).

For $a = -1$, equation (3.15) reads

$$F(1-c) + F(-c) = c(F(1-c) - F(c)) = F\left(\frac{c-1}{1+c}\right)\ . \tag{3.17}$$

Subtraction of ($\frac{1}{c}$ · (3.16) from (3.17) results in

(3.18) $$2F(-c) = \left(1 - \frac{1}{c}\right) F\left(\frac{c-1}{1+c}\right)$$

(3.19) $$F(a) = \frac{1}{2}\frac{1+a}{a} F\left(\frac{1+a}{a-1}\right)$$

$$F(\frac{1+a}{a-1}) = \frac{1}{2}\frac{1+\frac{1+a}{a-1}}{\frac{1+a}{a-1}} F\left(\frac{1+\frac{1+a}{a-1}}{\frac{1+a}{a-1}-1}\right)$$

(3.20) $$= \frac{a}{1+a} F(a) .$$

Comparison of (3.19) with (3.20) yields

$$F(a) = \frac{1}{2}\frac{a+1}{a}\frac{a}{1+a} F(a) .$$

Therefore one gets $F(a) = 0$ and $f(x) = \alpha x$ with $\alpha = \mathrm{const}$.

Result. Because of $g(x) = f(1-x) + g(1)$, we can write

$$g(x) = \Gamma_1 x - \Gamma_2 \quad (\Gamma_1, \Gamma_2 = \mathrm{const}) .$$

Therefore, the equation of a path through two points P, Q of order ϵ is in second order

$$\vec{x}(\epsilon\rho) = \vec{x}_P + \rho\,\vec{v} + \rho(\rho-1)\begin{pmatrix} \Gamma_1 v^1 v^2 - \Gamma_2\,(v^1)^2 \\ 0 \end{pmatrix} + O(\epsilon^3)$$

with $\vec{v} = \vec{x}_Q - \vec{x}_P$.

The parameter transformation $\rho = \sigma + \sigma(\sigma-1)\Gamma_2 v^1$ yields

$$\vec{x}(\epsilon\rho) = \vec{x}_P + \rho\,\vec{v} + \rho(\rho-1)\, v^1 v^2 \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} + O(\epsilon^3) .$$

The acceleration part can be eliminated by the transformation

$$x_1' = x^1 - \Gamma_1\, x^1 x^2 , x'^2 = x^2 - \Gamma_2\, x^1 x^2$$

(which leaves invariant the axes).

For the first coordinate we have

$$\begin{aligned} x'^1(\epsilon\rho) &= x^1(\epsilon\rho) - \Gamma_1 x^1(\epsilon\rho) x^2(\epsilon\rho) \\ &= (x_P^1 - \Gamma_1 x_P^1 x_P^2) \\ &\quad + \rho[(x_Q^1 - \Gamma_1 x_Q^1 x_Q^2) - (x_P^1 - \Gamma_1\, x_P^1 x_P^2)] + O(\epsilon^3) \\ &= x_P'^1 + \rho(x_Q'^1 - x_P'^1) + O(\epsilon^3) . \end{aligned}$$

With a similar proof for the second coordinate, the equation of a path through the points P, Q is in second order

$$\vec{x}'(\epsilon\rho) = \vec{x}'_P + \rho(\vec{x}'_Q - \vec{x}'_P) + O(\epsilon^3) .$$

4. EXTENSION TO HIGHER DIMENSIONS

4.1 Proof in 3 dimensions

We have proved up to now that there exists a chart $KS_{1,2}$ with the property that the acceleration part of an equation of a path through a point P on the x^1-axis and a point Q on the x^2-axis (both of order ϵ) vanishes. That means the equation is

$$\vec{x}(\epsilon\rho) = \vec{x}_P + \rho(\vec{x}_Q - \vec{x}_P) + O(\epsilon^3).$$

Equivalently there exist charts $KS_{2,3}$ and $KS_{1,3}$. Because the transformations, needed to get $KS_{i,j}$, didn't change the axes of the charts, the transformation which transforms $KS_{2,3}$ into $KS_{1,2}$ doesn't change the axes.

Let x^i be the coordinates of $KS_{1,2}$,

x'^i be the coordinates of $KS_{2,3}$;

then $x^i = \alpha^i_j x'^j + \Gamma^i_{jk}\, x'^j x'^k + \ldots$ is the transformation $KS_{2,3} \rightarrow KS_{1,2}$.

Because the transformation doesn't affect the axes, we have $\alpha^i_j = \delta^i_j$ and $\Gamma^i_{jj} = 0$ for $i \neq j$.

In $KS_{2,3}$ the equation of a path which passes through $P = \epsilon(0, p', 0)$ and $Q = \epsilon(0, 0, q')$ is

$$\vec{x}'(\epsilon\rho) = \epsilon(0, p', 0) + \epsilon\rho(0, -p', q') + O(\epsilon^3)\,. \tag{4.1}$$

Therefore in $KS_{1,2}$ the equation is

$$\begin{aligned}
x^i(\epsilon\rho) =& x'^i(\epsilon\rho) + \Gamma^i_{jk} x'^j(\epsilon\rho)\, x'^k(\epsilon\rho) \\
=& x'^i_P + \rho(x'^i_Q - x'^i_P) + O(\epsilon^3) \\
&+ \Gamma^i_{jk}[x'^j_P + \rho(x'^j_Q - x'^j_P) + O(\epsilon^3)] \\
&[x'^k_P + \rho\,(x'^k_Q - x'^k_P) + O(\epsilon^3)] \\
=& x'^i_P + \Gamma^i_{jk} x'^j_P x'^k_P \\
&+ \rho[(x'^i_Q - \Gamma^i_{jk}\, x'^j_Q x'^k_Q) - (x'^i_P + \Gamma^i_{jk}\, x'^j_P x'^k_P)] \\
&+ \rho(\rho - 1)\,\Gamma^i_{jk}(x'^j_Q - x'^j_P)(x'^k_Q - x'^k_P) \\
=& x^i_P + \rho\,(x^i_Q - x^i_P) \\
&+ \rho(\rho - 1)\,\Gamma^i_{jk}(x'^j_Q - x'^j_P)(x'^k_Q - x'^k_P) + O(\epsilon^3) \\
=& x^i_P + \rho(x^i_Q - x^i_P) \\
&+ \rho(\rho - 1)\,\Gamma^i_{jk}\,(x^j_Q - x^j_P)(x^k_Q - x^k_P) + O(\epsilon^3)\,,
\end{aligned} \tag{4.2}$$

because of

$$x'^i_Q - x'^i_P = x^i_Q - x^i_P + O(\epsilon^2)\ .$$

Finally the equation of a path through P and Q in $KS_{1,2}$ is

$$\begin{aligned}\vec{x}(\epsilon\rho) = {} & \vec{x}_P + \rho\,\vec{v} \\ & + \rho(\rho-1)\left[v^2 v^3 \begin{pmatrix}\Gamma^1_{23}\\ \Gamma^2_{23}\\ \Gamma^3_{23}\end{pmatrix} \right. \\ & \left. + (v^2)^2 \begin{pmatrix}0\\ \Gamma^2_{22}\\ 0\end{pmatrix} + (v^3)^2 \begin{pmatrix}0\\ 0\\ \Gamma^3_{33}\end{pmatrix}\right] \\ & + O(\epsilon^3)\end{aligned}$$

with

$$\vec{v} = \vec{x}_Q - \vec{x}_P = \left(0\,, -p' - \Gamma^2_{22}p'^2\,, -q' - \Gamma^3_{33}q'^2\right)$$

The parameter transformation $\rho = \sigma - \sigma(\sigma-1)(\Gamma^3_{33}v^3 + \Gamma^2_{22}v^2)$ yields

$$\vec{x}(\epsilon\rho) = \vec{x}_P + \sigma\vec{v} + \sigma(\sigma-1)v^2v^3 \begin{pmatrix}\tilde{\Gamma}^1_{23}\\ \tilde{\Gamma}^2_{23}\\ \tilde{\Gamma}^3_{23}\end{pmatrix} + O(\epsilon^3)$$

with

$$\begin{pmatrix}\tilde{\Gamma}^1_{23}\\ \tilde{\Gamma}^2_{23}\\ \tilde{\Gamma}^3_{23}\end{pmatrix} = \begin{pmatrix}\Gamma^1_{23}\\ \Gamma^2_{23} - \Gamma^3_{33}\\ \Gamma^3_{23} - \Gamma^2_{22}\end{pmatrix}\ .$$

In an equivalent way we see, that the equation of a path which passes through a point P (order ϵ) on the x^1-axis and a point Q (order ϵ) on the x^3-axis, has in second order in $KS_{1,2}$ the form

$$\vec{x}(\epsilon\rho) = \vec{x}_P + \rho\vec{v} + \rho(\rho-1)v^1v^3 \begin{pmatrix}\tilde{\Gamma}^1_{13}\\ \tilde{\Gamma}^2_{13}\\ \tilde{\Gamma}^3_{13}\end{pmatrix} + O(\epsilon^3)\ .$$

LEMMA 1. *In a chart* $KS'_{1,2}$ *defined by the transformation*

$$x'^i = x^i - \tilde{\Gamma}^i_{13} x^1 x^3 - \tilde{\Gamma}^i_{23} x^2 x^3$$

the equation of a path through two points P, Q *of order* ϵ *on any axes has in second order the form*

$$\vec{x}'(\epsilon\rho) = \vec{x}'_P + \rho(\vec{x}'_Q - \vec{x}'_P) + O(\epsilon^3) \ .$$

Proof: First of all we see, that the points $\vec{x}$ with $x^3 = 0$ are not affected by the transformation, so it is clear that the equation of a path through a point on the x^1-axis and a point on the x^2-axis is not affected by the transformation.

Now let us treat a path through a point P (order ϵ) on the x^1-axis and a point Q (order ϵ) on the x^3-axis.

Because the second component of the equation of this path is in $KS_{1,2}$

$$x^2(\epsilon\rho) = \rho(\rho-1)v^1 v^3 \tilde{\Gamma}^2_{13} + O(\epsilon^3) = O(\epsilon^2),$$

the part $\tilde{\Gamma}^i_{23} x^2(\epsilon\rho) x^3(\epsilon\rho) = O(\epsilon^3)$ does not affect the transformation of this path. Therefore the transformation of this path yields

$$\begin{aligned} x'^i(\epsilon\rho) =& x^i - \tilde{\Gamma}^i_{13} x^1(\epsilon\rho) x^3(\epsilon\rho) + O(\epsilon^3) \\ =& x^i_P + \rho\,(x^i_Q - x^i_P) + \rho(\rho-1)\tilde{\Gamma}^i_{13}\,(x^1_Q - x^1_P)(x^3_Q - x^3_P) \\ & - \tilde{\Gamma}^i_{13}(x^1_P + \rho(x^1_Q - x^1_P) + O(\epsilon^2)) \\ & (x^3_P + \rho(x^3_Q - x^3_P) + O(\epsilon^2)) \\ & + O(\epsilon^3) \\ =& x'^i_P + \rho\,(x'^i_Q - x'^i_P) + O(\epsilon^3) \ . \end{aligned}$$

The path through a point on the x^2 -axis and a point on the x^3-axis can be treated equivalently. ■

Finally we prove that in $KS'_{1,2}$ the equation of a path through any two points P and Q of order ϵ is in second order

$$\vec{x}'(\epsilon\rho) = \vec{x}'_P + \rho(\vec{x}'_Q - \vec{x}'_P) + O(\epsilon^3) \ .$$

In chapter 2 we derived that the general form of this equation is

$$\begin{aligned} \vec{x}'(\epsilon\rho) =& \vec{x}'_P + \rho(\vec{x}'_Q - \vec{x}'_P) \\ & + \epsilon^2\rho(\rho-1)\vec{B}\left(\frac{\vec{x}'_Q - \vec{x}'_P}{\epsilon}\right) + O(\epsilon^3) \ . \end{aligned}$$

Due the above Lemma, we see that $\vec{B}$ has the following property in $KS'_{1,2}$:

$$\vec{B}(0, u^2, u^3) = \vec{B}(u^1, 0, u^3) = \vec{B}(u^1, u^2, 0) = 0 \ ,$$

for all $u^1, u^2, u^3 \in \mathbb{R}$. Now we carry out Desargues' construction once again. We choose

$$p : x^1\text{-axis} \ ,$$
$$p' : x^2\text{-axis} \ ,$$
$$p'' : x^3\text{-axis}$$

and the points

$$A = \epsilon(-2u^1, 0, 0), \qquad B = \epsilon(-u^1, 0, 0)$$
$$A' = \epsilon(0, u^2, 0), \qquad B' = \epsilon(0, \frac{u^2}{3}, 0)$$
$$A'' = \epsilon(0, 0, -2u^3), \qquad B'' = \epsilon(0, 0, -\frac{u^3}{2}) \ .$$

Intersection of AA' and BB' :

$$\begin{aligned} AA' &: \vec{x}(\epsilon\rho) = [(-2u^1, 0, 0) + \rho(2u^1, u^2, 0)]\epsilon + O(\epsilon^3) \ , \\ BB' &: \vec{x}(\epsilon\sigma) = [(-u^1, 0, 0) + \sigma(u^1, \frac{u^2}{3}, 0)]\epsilon + O(\epsilon^3) \ . \end{aligned} \tag{4.3}$$

Intersection yields ($\rho = -1, \sigma = -3$)

$$e_1 = (-4u^1, -u^2, 0)\epsilon + O(\epsilon^3) \ .$$

Intersection of AA'' and BB'' :

$$AA'' : \vec{x}(\epsilon\rho) = [(-2u^1, 0, 0) + \rho(2u^1, 0, -2u^3)]\epsilon + O(\epsilon^3) \ .$$

$$BB'' : \vec{x}(\epsilon\sigma) = [(-u^1, 0, 0) + \sigma(u^1, 0, -\frac{u^3}{2})]\epsilon + O(\epsilon^3) \ .$$

Intersection yields ($\rho = -\frac{1}{2}$, $\sigma = -2$)

$$e_2 = (-3u^1, 0, u^3)\epsilon + O(\epsilon^3) \ .$$

Intersection of $A'A''$ and $B'B''$:

$$A'A'' : \vec{x}(\epsilon\rho) = [(0, u^2, 0) + \rho(0, -u^2, -2u^3)]\epsilon + O(\epsilon^3) \ .$$

$$B'B'' : \vec{x}(\epsilon\sigma) = [(0, \frac{u^2}{3}, 0) + \sigma(0, -\frac{u^2}{3}, -\frac{u^3}{2})]\epsilon + O(\epsilon^3) \ .$$

Intersection yields ($\rho = -2$, $\sigma = -8$)

$$e_3 = (0, 3u^2, 4u^3)\epsilon + O(\epsilon^3) .$$

Application of the Desargues condition:

The path through e_1 and e_2 has the equation

$$\vec{x}(\epsilon\rho) = \epsilon \begin{pmatrix} -4u^1 \\ -u^2 \\ 0 \end{pmatrix} + \epsilon\rho \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix} + \epsilon^2 \rho(\rho - 1) \begin{pmatrix} B^1(u^1, u^2, u^3) \\ B^2(u^1, u^2, u^3) \\ B^3(u^1, u^2, u^3) \end{pmatrix} + O(\epsilon^3).$$

If $u^2 = 0$ then $\vec{B} = 0$ (already proved). If $u^2 \neq 0$ then the parameter transformation

$$\rho = \sigma - \epsilon\sigma(\sigma - 1)\frac{B^2(u^1, u^2, u^3)}{u^2}$$

and the definitions $\tilde{B}^i := B^i - \frac{u^i}{u^2}B^2$ yield

$$\vec{x}(\epsilon\sigma) = \epsilon \begin{pmatrix} -4u^1 \\ -u^2 \\ 0 \end{pmatrix} + \epsilon\sigma \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix} + \epsilon^2 \sigma(\sigma - 1) \begin{pmatrix} \tilde{B}^1(u^1, u^2, u^3) \\ 0 \\ \tilde{B}^3(u^1, u^2, u^3) \end{pmatrix} + O(\epsilon^3) .$$

Now Desargues requires, that there exists a σ such that $\vec{e}_3 = \vec{x}(\epsilon\sigma)$. From the second component it follows that $\sigma = 4$. If we insert this result into the other components we get $\tilde{B}^1 = \tilde{B}^3 = 0$. Therefore $KS'_{1,2}$ represents a chart, in which the equation of a path through O with any direction fulfils, with a suitable choice of parameter,

(4.4) $$\frac{d^2\vec{x}}{d\tau^2}|_O = 0$$

But this is the standard definition of a geodesic path structure.

4.2 Remarks for the n-dimensional case

In this section no sum convention will be used.

For an arbitrary n-dimensional manifold a 2-dimensional subspace can be constructed in the same way as in section 2.2. In section 3 it was proved that a chart $KS_{1,2}$ can be chosen, in which the equation of a path connecting a point P on the x^1 -axis and a point Q on the x^2 -axis in order ϵ^2 has the form

$$\vec{x}(\epsilon\rho) = \vec{x}_P + \rho\,(\vec{x}_Q - \vec{x}_P) + O(\epsilon^3)\ , \tag{4.5}$$

where $\vec{x}$ is a general n-dimensional vector. Equivalently to calculations (4.1) and (4.2) it can be shown that in $KS_{1,2}$ the equation of a path connecting a point P on the x^i-axis and a point Q on the x^j-axis ($i < j \leq n$) has in second order the form

$$\vec{x}(\epsilon\rho) = \vec{x}_P + \rho\vec{v} + \rho(\rho-1)v^i v^j \begin{pmatrix} \tilde{\Gamma}^1_{ij} \\ \vdots \\ \tilde{\Gamma}^n_{ij} \end{pmatrix} + O(\epsilon^3)$$

with $\vec{v} = \vec{x}_Q - \vec{x}_P$.

In the chart $KS'_{1,2}$ defined by the transformation

$$x'^i := x^i - \sum_{k<l}^{n} \tilde{\Gamma}^i_{kl}\, x^k x^l\ ; \qquad \tilde{\Gamma}^i_{12} = 0\ ,$$

the equation of a path, connecting any two axes of the chart, has the form (4.5). (This is the generalization of Lemma 1.)

In order to prove with induction that in $KS'_{1,2}$ the equation of any path connecting two points in the ϵ-neighbourhood of O has the form (4.5), define the subspaces

$$P_{i_1,\dots,i_k} := \left\{\vec{x} \mid x^j = 0 \text{ for } j \neq i_1,\dots,i_k\right\} \quad k \leq n\,.$$

To prove that if for all $i_1,\dots,i_k \leq n$ the equation of a path connecting two points $P, Q \in P_{i_1,\dots,i_k}$ of order ϵ has the form (4.5), this also holds for a path connecting two points $P, Q \in P_{i_1,\dots,i_k,i_{k+1}}$ we carry out Desargues' construction once again. We choose the points

$$\begin{aligned} A &= \epsilon(0,\dots,-2u^{i_1},0,\dots,0,-2u^{i_2},0,\dots,\\ &\qquad 0,-2u^{i_{k-1}},0,\dots)\ ,\\ A' &= \epsilon(0,\dots,u^{i_k},0,\dots)\ ,\\ \text{and } A'' &= \epsilon(0,\dots,-2u^{i_{k+1}},0,\dots) \end{aligned}$$

and the paths $p = OA, p' = OA'$ and $p'' = OA''$. On p, p' and p'' we choose the points

$$B = \epsilon(0, \ldots, -u^{i_1}, 0, \ldots, 0, -u^{i_2}, 0, \ldots, 0, -u^{i_{k-1}}, 0, \ldots) ,$$

$$B' = \epsilon(0, \ldots, \frac{u^{i_k}}{3}, 0, \ldots) ,$$

$$\text{and } B'' = \epsilon(0, \ldots, -\frac{u^{i_{k+1}}}{2}, 0, \ldots) .$$

The calculation then runs equivalently to equations (4.3) up to (4.4).

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NOTE ADDED IN PROOF

A correspondence with J. Ehlers showed that some points in this paper deserve clarification:

a) In the Theorem (in the Introduction) we have to make explicit what it means that two paths meet up to corrections of order ϵ_0^3 : We consider only path structures with smoothness properties such that for any point $0 \in M$ there exists a neighbourhood U_0 in which the suitably parametrized paths through 0 induce a one-to-one exponential map $P = \exp(v)$ between the vectors $v \in T_0$ and the points $P \in U_0$. With one of the (equivalent) norms in T_0 we define $\epsilon_0 = \sup_{P,P' \in U_0} || \exp^{-1}(P) - \exp^{-1}(P')||$. Then two paths π_1, π_2 in U_0 meet up to corrections of order ϵ_0^3 if their minimal distance $d = \min_{Q_1 \in \pi_1, Q_2 \in \pi_2} || \exp^{-1}(Q_1) - \exp^{-1}(Q_2)||$ fulfils $d = O(\lambda^3)$ for $\epsilon_0 \rightarrow \lambda\epsilon_0$ and $\lambda \rightarrow 0$.

b) In the beginning of section 2.1 it is stated without proof that the acceleration field $\vec{A}$ fulfils $\vec{A}(\vec{x}, \lambda\vec{u}) = \lambda^2 \vec{A}(\vec{x}, \vec{u})$ for all $\lambda \neq 0$. A more precise statement would be that in the class of equivalent acceleration fields for a given path structure one can always find a field with this property. This can be proven in the following way (an alternative proof in the language of jet bundles is given in [13]):

A path may be given as the trace of the curve $\vec{x}(\tau)$, defined by the initial value problem $\vec{x}(0) = \vec{x}_0, \dot{\vec{x}}(0) = \vec{e}$ with $||\vec{e}|| = 1$, and $\ddot{\vec{x}} = 2\vec{A}(\vec{x}, \dot{\vec{x}})$, where $\cdot$ denotes the derivative with respect to τ. However, because a path is already uniquely defined by a point and a direction at this point, the curve $\vec{y}(\sigma)$, defined by the initial value problem $\vec{y}(0) = \vec{x}_0$, $\vec{y}\,'(0) = \lambda^{-1}\vec{e} = \lambda^{-1}\dot{\vec{x}}(0) = \vec{v}$, $\vec{y}\,''(\sigma) = 2\vec{A}(\vec{y}, \vec{y}\,')$ ($'$ denotes the derivative with respect to σ) has the same trace as $\vec{x}(\tau)$, i.e. there exists a parameter transformation $\sigma(\tau)$ such that $\vec{y}(\sigma(\tau)) = \vec{x}(\tau)$. Therefore $\ddot{\vec{x}} = \ddot{\vec{y}} = \vec{y}\,''\dot{\sigma}^2 + \vec{y}\,'\ddot{\sigma} = 2\vec{A}(\vec{y},\vec{y}\,')\dot{\sigma}^2 + \vec{y}\,'\ddot{\sigma} = 2\vec{A}(\vec{x},\dot{\vec{x}})$. Evaluating this relation at $\tau = 0$, it follows that there exists a function $f(\vec{x},\vec{v})$ with $||\vec{v}||^2\vec{A}(\vec{x}_0,\vec{v}/||\vec{v}||) - \vec{A}(\vec{x}_0,\vec{v}) = \frac{1}{2}\vec{v}f(\vec{x}_0,\vec{v})$. Then the acceleration field $\tilde{\vec{A}}(\vec{x},\vec{v}) := ||\vec{v}||^2\vec{A}(\vec{x},\vec{v}/||\vec{v}||)$ fulfils $\tilde{\vec{A}}(\vec{x},\lambda\vec{v}) = \lambda^2\tilde{\vec{A}}(\vec{x},\vec{v})$, and $\tilde{\vec{A}}$ represents the same path structure as $\vec{A}$, because the curve $\vec{y}(\sigma(\tau))$ defined by $\vec{y}(0) = \vec{x}_0, \vec{y}\,'(0) = \vec{v},\ \vec{y}\,'' = 2\tilde{\vec{A}}(\vec{y},\vec{y}\,')$ and $\sigma(0) = 0, \dot{\sigma}(0) = 1,\ \ddot{\sigma} = -\dot{\sigma}f(\vec{y}(\sigma),\dot{\sigma}\vec{y}\,'(\sigma))$ is identical to the curve $\vec{x}(\tau)$ defined by $\vec{x}(0) = \vec{x}_0, \dot{\vec{x}}(0) = \vec{v},\ \ddot{\vec{x}} = 2\vec{A}(\vec{x},\dot{\vec{x}})$.

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